CIRCULATION FLOW OF AN IDEAL LIQUID NEAR WALLS

I. A. Belov

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§1. It is well known that in the case of the flow of accelerated streams near stationary boundaries the effect of viscosity on the flow field appears at very small distances from the boundaries of the region being discussed. In a significant portion of the region the flow may be taken to be ideal and irrotational or rotational with the strength of the vortex determined by the conditions of formation of the stream far from the boundaries of the region.

Let us consider in the Cartesian coordinate system xoy a two-dimensional flow of an incompressible liquid near solid walls or the symmetry plane $x=0$ and $y=0$ with the point of flow stagnation being $x=y=0$. Let x_∞ and y_∞ be the specified cross sections with fixed boundary conditions subject to determination. Let us seek a solution for the flow velocity components in the form

$$v = -F(y)\Phi'(x), \quad u = F'(y)\Phi(x), \tag{1.1}$$

where v and u are the velocity components along the normal and along the tangent to the surface $y=0$, referred to the characteristic velocity at the boundary y_∞. The obvious boundary conditions for the problem under discussion are

$$v(x, 0) = 0, \; F(0) = 0; \; u(0, y) = 0; \; \Phi(0) = 0. \tag{1.2}$$

It follows from (1.1) that with the boundary conditions (1.2) the velocity flux through the surface bounding the region under discussion is equal to zero for any values of the functions F and Φ at the boundaries y_∞ and x_∞. We assume that $\max|v(x, y_\infty)| = \max|u(x_\infty, y)|$. From this we obtain from (1.1), excluding the conditions $F(y_\infty) = \Phi(x_\infty) = 0$, which give trivial solutions,

$$\Phi(x_\infty) = F(y_\infty)\Phi'_m/F'_m, \tag{1.3}$$

where

$$\Phi'_m/F'_m = \text{const}, \; (\Phi', F')_m = \max(\Phi', F').$$

Without losing the generality of the discussions, we adopt $x_\infty = y_\infty = 1$, where the length of the side of the square $x, y \in [0, 1]$ is used as the characteristic length. The condition of the absence of a vortex in the flow is written in the form

$$F\Phi(F''/F + \Phi''/\Phi) = 0.$$

Excluding the trivial solutions $F = \Phi = 0$, we obtain

$$F''/F = -\Phi''/\Phi = C. \tag{1.4}$$

For $C=0$ there follows from (1.4) the well-known solution for a uniform flow in the vicinity of the stagnation point [1] $F = C_1 y$, $\Phi = C_2 x$; $v = -by$, $u = bx$, where $C_1 = F(1)$; $C_2 = \Phi(1)$; and $b = C_1C_2$ is the gradient of the velocity at the stagnation point.

For $C < 0$ the solutions (1.4) have the form $F = C_1 \sin\sqrt{-C}y$, $\Phi = C_2 \sinh\sqrt{-C}x$, where $C_1 = F(1)/\sin\sqrt{-C}$ and $C_2 = \Phi(1)/\sinh\sqrt{-C}$.

In this case we obtain from (1.3) the following equation for the determination of the values of $\sqrt{-C}$:

$$\sin\sqrt{-C}\cdot\operatorname{ch}\sqrt{-C}x_m/(\operatorname{sh}\sqrt{-C}\cdot\cos\sqrt{-C}y_m) = 1,$$

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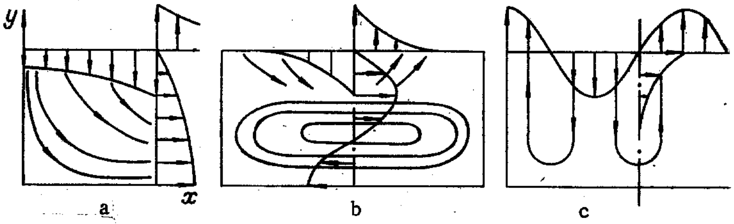

Fig. 1

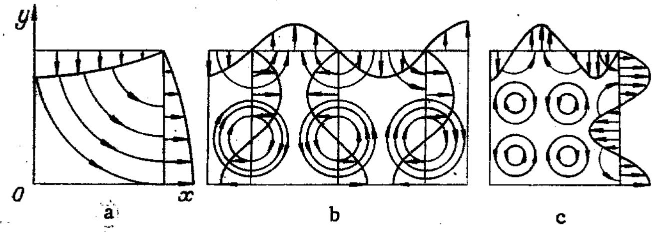

Fig. 2

where x_m and y_m are the cross sections at which $\Phi' = \Phi'_m$ and $F' = F'_m$, respectively. We note that the roots of this equation agree in size with the roots of the equation $\Phi'_m / F'_m = 1$ in the case in which $F(1) = \Phi(1)$. Having adopted in the following $C_1 = a$, where $a = F(1) = \Phi(1)$, we determine the constants $\sqrt{-C}$ and C_2 as

$$\sqrt{-C} = (-1)^k (2k+1)\pi/2, \quad C_2 = (-1)^k a / \mathrm{sh}(2k+1)\pi/2, \quad k = 0, 1, \ldots,$$

which satisfy (1.3) to a high degree of accuracy for $k > 0$. For selected values of $\sqrt{-C}$, C_1, and C_2 the solutions for F and Φ are of the form

$$F = (-1)^k a \sin(2k+1)(\pi/2)y, \quad \Phi = a\,\mathrm{sh}(2k+1)(\pi/2)x / \mathrm{sh}(2k+1)\pi/2.$$

Similar expressions are obtained for $C > 0$:

$$F = a\,\mathrm{sh}(2k+1)(\pi/2)y / \mathrm{sh}(2k+1)\pi/2, \quad \Phi = (-1)^k a \sin(2k+1)(\pi/2)x.$$

The stream function of the flow in question has the form

$$\psi = (-1)^k a^2\,\mathrm{sh}(2k+1)(\pi/2)z \sin(2k+1)(\pi/2)\zeta / \mathrm{sh}(2k+1)\pi/2,$$
$$z \equiv x, \ \zeta \equiv y, \ C < 0; \ z \equiv y, \ \zeta \equiv x, \ C > 0$$

for $C \neq 0$. We note that the sums of the solutions of the type indicated above for $C = 0$ and $C \gtrless 0$ are also solutions of Eq. (1.4) with the boundary conditions (1.2) and (1.3). Let us investigate particular solutions of (1.4) for $C \neq 0$. The boundary conditions on the surfaces $x = y = 1$ on the assumption that $a^2 = a_k^2 = \tanh(2k+1)\frac{\pi}{2} \Big/ (2k+1)\frac{\pi}{2}$ are of the form

$$v(x, 1) = -\mathrm{ch}(2k+1)(\pi/2)x / \mathrm{ch}(2k+1)\pi/2, \quad u(x, 1) = 0,$$
$$v(1, y) = (-1)^{k+1} \sin(2k+1)(\pi/2)y,$$
$$u(1, y) = (-1)^k\,\mathrm{th}(2k+1)(\pi/2) \cos(2k+1)(\pi/2)y$$

for sufficiently large values of k and for $C < 0$, where the maximum velocity at the boundary $y = 1$ is $v(1, 1) = 1$; and the ratio of $v(1, 1)$ to the minimum velocity on the same boundary $v(0, 1)$ is equal to $\cosh(2k+1)\pi/2$. Similar relations are obtained for $C > 0$. Having taken the same value for a^2 as in the preceding case, the boundary conditions on the surfaces $x = y = 1$ are written in the form

$$v(x, 1) = (-1)^{k+1}\,\mathrm{th}(2k+1)(\pi/2) \cos(2k+1)(\pi/2)x,$$
$$u(x, 1) = (-1)^k \sin(2k+1)(\pi/2)x, \quad v(1, y) = 0,$$
$$u(1, y) = \mathrm{ch}(2k+1)(\pi/2)y / \mathrm{ch}(2k+1)\pi/2$$

for sufficiently large values of k, where the maximum velocity on the boundary $x = 1$ is $u(1, 1) = 1$, and the ratio of $u(1, 1)$ to the minimum velocity on this boundary $u(1, 0)$ is equal to $\cosh(2k+1)\pi/2$.

It follows from a comparison of the solutions obtained that when $C < 0$ the velocities $v(x, 1)$, $u(x, 1)$, $u(1, y)$, and $v(1, y)$ are equal in absolute value to the velocities $u(1, y)$, $v(1, y)$, $v(x, 1)$, and $u(x, 1)$, respectively, i.e., the solution for $C < 0$ in the region $x, y \in [0, 1]$ is coupled with the solution for $C > 0$ in the region $x \in [1, 2], y \in [0, 1]$, and for $C > 0$ in the region $x, y \in [0, 1]$ – with the solution for $C < 0$ in the region $x \in [1, 2], y \in [0, 1]$. The possibility of coupling of the solutions is provided for by the choice as the boundary conditions on the surfaces $x = y = 1$ of the periodicity conditions of the flow, whose appearance is caused by the solid wall or the symmetry plane (the plane of interaction of the two flows) located beneath the flow. This will be the plane $x = 2$ for the region $x, y \in [0, 1]$; for the region $x \in [1, 2], y \in [0, 1)$ – the $y = 2$ plane, and so on. When $a_0 \neq 0$ and $a_k = 0$ ($k = 1, 2, \ldots$) (Fig. 1a), the flow in the region under discussion is accomplished without the formation of circulation zones near the walls, and when $a_0 = 0$ and $a_k \neq 0$ ($k = 1, 2, \ldots$) – with circulation zones (the $\psi = 0$ lines intersect with the surfaces $x = y = 1$ at the points $x_0, y_0 = 2n/(2k + 1)$; $x_0, y_0 \leq 1$, $k = 1, 2, \ldots$, $n = 1, 2, \ldots, k$). The number of circulation zones corresponds to the selected value of k; when $C < 0$ and $k = 1$ (Fig. 1b) the circulation zone for the flow in the region $x, y \in [0, 1]$ is oriented along the wall $y = 0$, and when $C > 0$ and $k = 1$ (Fig. 1c), the circulation zone for the flow in this same region is oriented along the normal to the wall $y = 0$.

§2. Let the vorticity in the flow of an ideal liquid be nonzero in the region $x, y \in [0, 1]$. In the present formulation the amount of vorticity is determined by the effects of the viscosity of the liquid outside the boundaries of the region under discussion. As earlier for irrotational flow, adopting dependences of v and u of the form (1.1), which guarantee fulfillment of the Stokes' theorem, we obtain from the vorticity transport equation

$$(FF''' - F'F'')/FF' = (\Phi\Phi''' - \Phi'\Phi'')/\Phi\Phi' = C. \tag{2.1}$$

The solutions of (2.1) in the case of the boundary conditions

$$F(0) = \Phi(0) = 0,\ F(1) = \Phi(1) = a,\ F'(1) = \Phi'(1) = d \tag{2.2}$$

were investigated in [2] for $C \lessgtr 0$. We will show that in the general case one should take account of the solutions which describe circulation flows of a liquid in the region under discussion in addition to those obtained in [2]. When $C = 0$, it follows from (2.1) that

$$\varphi'^2 = C_2 + C_1\varphi^2,\ C_2 = d^2 - C_1a^2,\ C_1 = \varphi''(1),\ \varphi \equiv F,\ \Phi.$$

When $P = C_1a^2/(C_1a^2 - d^2) \lessgtr 0$, the solution for φ is of the form

$$\begin{gathered} \varphi/a = z,\ a = d,\ P = 0; \\ \varphi/a = \sqrt{(C_1a^2 - d^2)/C_1a^2}\cdot\sin\sqrt{-C_1}z,\ P > 0; \\ \varphi/a = \sqrt{(C_1a^2 - d^2)/C_1a^2}\cdot\operatorname{sh}\sqrt{C_1}z,\ P < 0,\ z \equiv x,\ y, \end{gathered} \tag{2.3}$$

i.e., when $C_1 \leq 0$ and $P > 0$, we have an unbounded set of real solutions for φ, and when $C_1 \geq 0$ and $P < 0$, an infinite number of real solutions is attained only when $d^2 \geq C_1a^2$. The quantity C_1 depends on the ratio $d/a = \varphi'(1)/\varphi(1)$, and for the conditions (2.2) it is determined from

$$\begin{gathered} d^2 = -C_1a^2\operatorname{ctg}^2\sqrt{-C_1},\ P > 0, \\ d^2 = C_1a^2\operatorname{cth}^2\sqrt{C_1},\ P < 0. \end{gathered} \tag{2.4}$$

If $\varphi'(1) = d$ is taken equal to zero, then when $P > 0$ the roots of Eq. (2.4) will be $\sqrt{-C_1} = (-1)^k(2k + 1)\pi/2$, $k = 0, 1, \ldots$ for negative values of C_1; when $P < 0$ and $d = 0$, there are no real roots $\sqrt{C_1}$ of Eq. (2.4) on the positive part of the number axis. When $d/a = 1$, $\sqrt{-C_1} = 0$, 4.4934, 7.7253, 10.9041, 14.0662, and so on, but $\sqrt{C_1} = 0$. When $C_1 = 0$, the solution (2.4) gives $d/a = 1$, which corresponds to the case analyzed above of the flow of an irrotational stream in the neighborhood of the stagnation point.

When $d^2 < C_1a^2$, we have a complex solution for φ. Setting $\sqrt{C_1} = p + iq$, we obtain from (2.4) two real equations for p and q:

$$p\operatorname{cth}p + p\operatorname{th}p = q\operatorname{tg}q - q\operatorname{ctg}q = C_3, \tag{2.5}$$

where C_3 is some constant which determines the value of the ratio d/a for each pair of roots (p, q).

It is obvious from (2.5) that one of the values of the number p corresponds to k different values of the number q. For larger values of p the constant $C_3 \to 2p$, and $q \to (2k + 1)\pi/2$, $k = 0, 1, \ldots$. Let us rewrite (2.5) in the form

$$\operatorname{th}^2 p = -\frac{d/a + q/\operatorname{tg}q}{d/a + q\operatorname{tg}q}\ (d/a > 0);\ \operatorname{th}^2 p = \frac{d/a - q/\operatorname{tg}q}{q\operatorname{tg}q - d/a}\ (d/a < 0).$$

It follows from the last relations that solutions (p, q) occur for $d/a \lessgtr 0$ and $\tan q < 0$ with q taken from the range:

$$\mp(d/a)/\mathrm{tg}\, q < q < \mp (d/a)\, \mathrm{tg}\, q;\ \mp (d/a)\, \mathrm{tg}\, q < q < \mp (d/a)/\, \mathrm{tg}\, q.$$

When $d/a = \pm 1\ \pi/2 < q < 2$; $3\pi/2 < q < 4.9$, $2.8 < q < \pi$, $6.12 < q < 2\pi$, and so on. Since $0 \le p < \infty$, let us determine the boundaries with respect to d/a of the existence of complex solutions for φ. When $p \to \infty$, $d/a = \pm q(\tan^2 q + 1)/(2 \tan q)$; when $p=0$ $d/a = \pm q/\tan q$. If one takes $q=0$, then when $p \to \infty$, $d/a = \pm 0.5$, and when $p=0$, $d/a = \pm 1$. The last limit corresponds to $C_1 = 0$, and consequently the condition $d^2/a^2 < C_1$ of the existence of complex solutions of (2.3) is not fulfilled. It is possible to obtain that in this case the quantity p is located in the range $0.725 \le p < \infty$.

The solution for φ when $\sqrt{C_1} = p + iq$ is of the form

$$\varphi/a = [(Ap + Bq) + i(Bp - Aq)]\, \mathrm{sh}\, pz \cos qz + [(Aq - Bp) + i(Ap + Bq)]\, \mathrm{ch}\, pz \sin qz,$$

where

$$A = \frac{(\alpha + \sqrt{\alpha^2 + \beta^2})^{1/2}}{(p^2 - q^2)\sqrt{2}};\ B = \frac{(-\alpha + \sqrt{\alpha^2 + \beta^2})^{1/2}}{(p^2 - q^2)\sqrt{2}},$$
$$\alpha = d^2/a^2 - p^2 + q^2,\ \beta = -2pq.$$

We note that just as for irrotational flow the sums of the solutions in question are, in the case of real $C_1 \gtrless 0$, $C_1 = 0$, and complex values of C_1, solutions of Eq. (2.3) with the boundary conditions (2.2). Let us investigate particular solutions of (2.3). Having taken $a^2 = \sin^2\sqrt{-C_1}/\sqrt{-C_1}$, we obtain

$$v = -\sin\sqrt{-C_1}y \cos\sqrt{-C_1}x,\ u = \sin\sqrt{-C_1}x \cos\sqrt{-C_1}y$$

for $C_1 < 0$, where $C_1 = C_1(a, d)$ is found from (2.4). If $d=0$, then $\sqrt{-C_1} = (-1)^k(2k+1)\pi/2$, $k = 0, 1, \ldots$, and then for different values of k the flow in the region $x, y \in [0, 1]$ under discussion represents a flow associated with the turning by a right angle of a nonuniform stream oriented along the normal to the plane $y=0$ without the formation of circulation zones ($k=0$, Fig. 2a) and with circulation zones ($k=1$, Fig. 2b; $k=2$, Fig. 2c). The appearance of circulation zones within the borders of the region under discussion, just as for irrotational flow, which is caused by the periodicity conditions of the flow on the boundaries $x=y=1$. The number of circulation zones (vortices) and the distance between the centers of adjacent vortices are equal to k^2 and $2/(2k+1)$, respectively; the distance between the planes $x, y=0$ and the center of the vortex nearest to the stagnation point is equal to $x, y = 1/(2k+1)$.

For $C_1 > 0$ and $d^2 > C_1 a^2$, having taken $a^2 = \sinh^2\sqrt{C_1}/\sqrt{C_1}$, we obtain

$$v = -\mathrm{sh}\sqrt{C_1}y\, \mathrm{ch}\sqrt{C_1}x,\ u = \mathrm{ch}\sqrt{C_1}y\, \mathrm{sh}\sqrt{C_1}x,$$

where $C_1 = C_1(a, d)$ is found from (2.4).

The given solution describes the flow associated with the turning by a right angle of a nonuniform stream with a specified twisting law of the stream in the cross sections $x=y=1$ in the flow plane.

For $d^2 < C_1 a^2$, having taken $a^2 = (p^2 - q^2)/(p^2 + q^2)$, we obtain

$$\mathrm{Re}\,(v) = \mathrm{ch}\, py\, \mathrm{ch}\, px \cos qy \cos qx[\mathrm{tg}\, qx\, \mathrm{th}\, px(M\, \mathrm{tg}\, qy - N\, \mathrm{th}\, py) - (M\, \mathrm{th}\, py + N\, \mathrm{tg}\, qy)];$$

$$\mathrm{Re}\,(u) = \mathrm{ch}\, py\, \mathrm{ch}\, px \cos qy \cos qx[\mathrm{tg}\, qy\, \mathrm{th}\, py(N\, \mathrm{th}\, px - M\, \mathrm{tg}\, qx) + (M\, \mathrm{th}\, px + N\, \mathrm{tg}\, qx)],$$

where p and q are found from (2.5); $M = p\sqrt{\alpha^2 + \beta^2} + \beta q$; and

$$N = q\sqrt{\alpha^2 + \beta^2} - \beta p,\ \alpha = d^2(p^2 + q^2)/(p^2 - q^2)^2 - p^2 + q^2,$$
$$\beta = -2pq.$$

The solution given is intermediate between the solutions obtained for $C_1 < 0$ and $C_1 > 0$. It is evident from the relations given for v and u that when $p=0$ the complex solution of (2.3) has the same form as does the real solution for $C_1 < 0$; when $q \to (2k+1)\pi/2$, $k = 0, 1, \ldots$, the solution of (2.3) changes into the real solution for $C_1 > 0$.

§3. The solutions derived represent flows of an ideal liquid near an obstacle, which require for their practical realization a corresponding restriction of the stream, including the satisfaction of vorticity conditions, at a specified distance from the obstacle. Rotational flows are of considerable interest. Flows near an obstacle with protruding separating partitions (flows in impactors) and flows arising in ventilation systems, laminated heat exchangers, and similar structures serve as physical analogs of the solutions obtained in this case. Flows of this type occur upon the interaction of a number of discrete jets with an obstacle, when the stream is repeatedly slowed down on fixed surfaces parallel to the obstacle, as, for example, in the case of jet cooling of the interior cavities of turbine vanes or in the case of the motion of a liquid under the body of vertical-takeoff and -landing aircraft and air-cushion devices.

We note some distinctive features of the solutions obtained with application to the latter problem.

For irrotational flow the choice of the value of the parameter C in Eq. (1.4), and consequently, the type of boundary conditions on the surfaces $x = y = 1$, determines the value of the ratio of the maximum and minimum flow velocities on these surfaces. Having taken the maximum value of the velocity as unity, we obtain that for $C < 0$ ($k = 1, 2, \ldots$) this ratio $v(1, 1)/v(0, 1)$ is equal to $\cosh(3\pi/2)$, $\cosh(5\pi/2)$, and so on. Thus for the occurrence of k circulation zones at the surface of the barrier it is necessary in this case to produce a flow which overflows onto the obstacle with a maximum circumferential velocity, $v(1, 1)$, which exceeds by a factor of $\cosh(2k+1)(\pi/2)/\cosh(\pi/2)$ the maximum velocity in the external flow; when the motion along the obstacle occurs without the formation of circulation zones (for $a_0 \neq 0$ and $a_k = 0$, $k = 1, 2, \ldots$) the ratio $v(1, 1)/v(0, 1) \simeq \cosh(\pi/2)$. The velocity distribution with a circumferential maximum is a typical distribution which arises at the truncation of a nozzle upon the interaction of the jet with an obstacle within the borders of the initial section of the jet. The application of discrete jets at small distances of the nozzle truncation from an obstacle and for a ratio of the maximum and minimum velocities at the nozzle truncation of the order of $\cosh(3\pi/2)$ can, as follows from the present analysis, result in developed rotational motion of the liquid beneath the surface of the nozzle truncation.

Just as for rotational flow, vortex flow near an obstacle is determined by the type of boundary conditions at the surfaces $x = y = 1$ and, as follows from the present analysis, depends on the choice of the value of the parameter C_1 in Eqs. (2.3), which characterizes the strength of the vortex in the external flow. Using the solutions obtained for periodic conditions on the surfaces $x = (2k+1)$, $k = 0, 1, 2, \ldots$, for the description of the flow which occurs in the case of the overflowing along the normal to the obstacle of a number of nonparallel jets, we obtain that for a stream in the cross section $y = 1$ with zero velocity component in the direction of the obstacle ($d = 0$) and for $C = 0$ and $C_1 < 0$, the strength of the vortex on the surface $y = 1$ is equal to $\Omega(x, 1) = 2\sqrt{-C_1}\sin\sqrt{-C_1}x$, where $\sqrt{-C_1} = (-1)^k(2k+1)\pi/2$. As k increases, the maximum value of $\Omega(x, 1)$ increases by a factor of $(2k+1)$ in comparison with rotational flow. The choice of a sufficiently large number of vortices permits in this case representing the flow in the region under discussion as some model "turbulent" flow consisting of a system of vortex particles. One can note that a system of k^2 vortices was used in the case of $C = d = 0$ and $C_1 < 0$ in [3] for the imitation of the turbulence of free flow in the case of flow around a plate in the longitudinal direction.

LITERATURE CITED

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